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Electromagnetic depolarization dyadics and elliptic integrals

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Abstract. Depolarization dyadics are essential for the characterization of electromagnetic fields in source regions. As such they are key ingredients in the formulation of homogenization theories of composite media. Explicit expressions for the depolarization dyadics for a biaxial dielectric anisotropic medium (encompassing orthorhombic, monoclinic and triclinic crystallographic classes) are presented here. The results are expressed in terms of elliptic functions of the first and second kind.

1. Introduction

To find the solution to the problem of scattering of an electromagnetic wave by a body of a certain geometry with given material parameters is generally very difficult. The mathematical formulation leads to fully three-dimensional integral equations which can, in general, only be solved numerically. Quite often, however, it is sufficient to construct approximative solutions which apply in a certain regime only. The far-field approximation is perhaps most well known in that context. At the other end of the scale is the near-field structure that can provide the required information in certain applications.

Within the last decade, a significant proportion of electromagnetics research has concentrated on materials: to establish novel effects generated by the use of naturally occurring or manufactured materials in electromagnetic applications; see the proceedings of specialist conferences for further information [1–3]. A widely used approach to the conceptualization of complex materials is through composite media. A composite is usually formed by combining two or more constituent media which in itself are homogeneous and described by certain constitutive (material) parameters. The aim of homogenization theories is to derive (or at least provide *estimates* for) constitutive parameters of the (homogenized) composite that depend on those of the constituent media. In the first instance this calculation requires the determination of the scattering response of an individual object (of a given shape and made of a certain material) illuminated by an electromagnetic wave.

A method commonly used to establish the scattering response of electrically small objects is the *long-wavelength approximation*. In the derivation of this estimate, two assumptions have to be made: (i) the current density distributions which are induced inside the object are uniform throughout its volume, and (ii) the volume itself is electrically small. These assumptions permit a replacement of the actual scattering object by electric and

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magnetic dipoles. The electromagnetic problem is thus reduced to finding the near-field of these dipoles—a procedure commonly known as obtaining the field in the *source region*.

In a further simplification (by simply neglecting certain terms), the long-wavelength approximation yields the so-called *Rayleigh estimate* of the electromagnetic field in the source region as follows†:

$$\boldsymbol{E}(\boldsymbol{x}) \cong \boldsymbol{E}_h(\boldsymbol{x}) + \underline{\underline{D}}_{\mathrm{ee}}(\boldsymbol{x}) \cdot \boldsymbol{J}_e(\boldsymbol{x}) + \underline{\underline{D}}_{\mathrm{em}}(\boldsymbol{x}) \cdot \boldsymbol{J}_m(\boldsymbol{x}) \tag{1}$$

$$\boldsymbol{H}(\boldsymbol{x}) \cong \boldsymbol{H}_h(\boldsymbol{x}) + \underline{\underline{D}}_{\mathrm{me}}(\boldsymbol{x}) \cdot \boldsymbol{J}_e(\boldsymbol{x}) + \underline{\underline{D}}_{\mathrm{mm}}(\boldsymbol{x}) \cdot \boldsymbol{J}_m(\boldsymbol{x}). \tag{2}$$

Here, $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electromagnetic field phasors (a harmonic time-dependence of $\exp(-\mathrm{i}\omega t)$, where ω is the circular frequency, is assumed throughout and suppressed henceforth). $\boldsymbol{E}_h$, $\boldsymbol{H}_h$ comprise the complementary function (i.e. they are a solution of the homogeneous version of Maxwell's equations) and while they play no further role here, they are included due to their importance for the solution of the actual scattering problem. Finally, $\boldsymbol{J}_e$ and $\boldsymbol{J}_m$ are the electric and the magnetic current densities, respectively.

There are altogether four depolarization dyadics in equations (1) and (2): they are usually referred to as of electric ($\underline{\underline{D}}_{\mathrm{ee}}$), magnetic ($\underline{\underline{D}}_{\mathrm{mm}}$) and magnetoelectric or mixed ($\underline{\underline{D}}_{\mathrm{em}}$, $\underline{\underline{D}}_{\mathrm{me}}$) type‡. A concise expression for the depolarization dyadic $\underline{\underline{D}}_{\mathrm{ee}}$ of a general anisotropic dielectric medium was first given in [5] (for that particular case $\underline{\underline{D}}_{\mathrm{ee}}$ is the only nontrivial of the four depolarization dyadics). Further generalization focused on the most general, linear, homogeneous medium, a so-called *bianisotropic* medium, characterized by frequency-domain constitutive relations

$$\boldsymbol{D}(\boldsymbol{x}) = \underline{\underline{\epsilon}} \cdot \boldsymbol{E}(\boldsymbol{x}) + \underline{\underline{\xi}} \cdot \boldsymbol{H}(\boldsymbol{x}) \tag{3}$$

$$\boldsymbol{B}(\boldsymbol{x}) = \underline{\underline{\zeta}} \cdot \boldsymbol{E}(\boldsymbol{x}) + \underline{\underline{\mu}} \cdot \boldsymbol{H}(\boldsymbol{x}) \tag{4}$$

wherein the permittivity dyadic $\underline{\underline{\epsilon}}$, the permeability dyadic $\underline{\underline{\mu}}$ and the magnetoelectric dyadics $\underline{\underline{\xi}}$ and $\underline{\underline{\zeta}}$ are in general complex-valued and functions of the circular frequency ω.

Expressions for the depolarization dyadics pertaining to a cavity of given shape in a general bianisotropic medium were obtained in [6] and are given by

$$\underline{\underline{D}}_{\lambda\lambda'} = \frac{1}{4\pi\,\mathrm{i}\omega} \int_{\phi=0}^{2\pi} \mathrm{d}\phi \int_{\theta=0}^{\pi} \mathrm{d}\theta\ \sin\theta \frac{(\hat{\boldsymbol{x}} \cdot \underline{\underline{\tau}}_{\lambda\lambda'} \cdot \hat{\boldsymbol{x}})\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{(\hat{\boldsymbol{x}} \cdot \underline{\underline{\epsilon}} \cdot \hat{\boldsymbol{x}})(\hat{\boldsymbol{x}} \cdot \underline{\underline{\mu}} \cdot \hat{\boldsymbol{x}}) - (\hat{\boldsymbol{x}} \cdot \underline{\underline{\xi}} \cdot \hat{\boldsymbol{x}})(\hat{\boldsymbol{x}} \cdot \underline{\underline{\zeta}} \cdot \hat{\boldsymbol{x}})} \tag{5}$$

$(\lambda, \lambda' = e, m)$, where the abbreviations

$$\underline{\underline{\tau}}_{\mathrm{ee}} = \underline{\underline{\mu}} \qquad \underline{\underline{\tau}}_{\mathrm{em}} = -\underline{\underline{\xi}} \qquad \underline{\underline{\tau}}_{\mathrm{me}} = -\underline{\underline{\zeta}} \qquad \underline{\underline{\tau}}_{\mathrm{mm}} = \underline{\underline{\epsilon}} \tag{6}$$

were used. The vector $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\theta, \phi)$ is the radial unit vector in a spherical coordinate system located at the centre of the (exclusion) region which is assumed to be spherical; the integration in (5) is over the surface of that sphere.

Without restricting the parameters in the constitutive dyadics, (5) will in general have to be evaluated numerically. As far as closed-form evaluations of (5) for a spherical geometry are concerned, they can be trivially obtained for isotropic [4] and bi-isotropic [7] media (all constitutive dyadics are proportional to the unit dyadic $\underline{\underline{I}}$). The simplest example for an anisotropic medium is a uniaxial one: at least one of the constitutive dyadics is of the form $\underline{\underline{a}} = a_t \underline{\underline{I}} + (a_z - a_t)\boldsymbol{u}_z\boldsymbol{u}_z$ where $\boldsymbol{u}_z$ is a unit vector. Explicit results for a uniaxial dielectric medium are available [5]: generalizations to a uniaxial dielectric–magnetic medium [8, 9], and to a uniaxial bianisotropic medium [10] have become available recently and further

† Vectors are in bold and dyadics are underlined twice, while · symbolizes a dot-product.

‡ It is noted that in the dielectric *isotropic* case, the term *depolarization dyadic* is reserved for a dimensionless quantity, see [4]. The depolarization dyadics defined here have different physical dimensions.

extensions to an affinely transformable axially uniaxial bianisotropic medium have been published [11]. In the context of affine transformations, mention should also be made of results pertaining to a simple symmetric bianisotropic medium [12]. No closed-form results beyond (either anisotropic and bianisotropic) uniaxiality for truly dispersive media have been reported to date.

Here, a *biaxial* dielectric anisotropic medium is considered. The constitutive dyadics are given by

$$\underline{\underline{\epsilon}}_{\rm bi} = \epsilon_0 \underline{\underline{\hat{\epsilon}}}_{\rm bi} = \epsilon_0(\epsilon_x \boldsymbol{u}_x\boldsymbol{u}_x + \epsilon_y \boldsymbol{u}_y\boldsymbol{u}_y + \epsilon_z \boldsymbol{u}_z\boldsymbol{u}_z) \qquad \underline{\underline{\mu}} = \mu_0 \underline{\underline{I}} \qquad \underline{\underline{\xi}} = \underline{\underline{\zeta}} = \underline{\underline{0}} \tag{7}$$

where $\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z$ is the triad of unit vectors in a Cartesian coordinate system and μ_0 is the permeability of free space. In crystallographic applications, the dyadic $\underline{\underline{\hat{\epsilon}}}_{\rm bi}$ may be used to characterize many representatives from the orthorhombic, monoclinic and triclinic crystallographic classes [13]. It is essential here to realize that in order to describe realistic materials, the constitutive parameters ϵ_x, ϵ_y, ϵ_z must be complex-valued.

The vanishing of the magnetoelectric dyadics $\underline{\underline{\xi}}$ and $\underline{\underline{\zeta}}$ immediately entails: $\underline{\underline{D}}_{\rm em} = \underline{\underline{D}}_{\rm me} = \underline{\underline{0}}$, whereas $\underline{\underline{D}}_{\rm mm}$ reduces trivially to the expression for an isotropic medium [4]. The aim of this communication is to derive $\underline{\underline{D}}_{\rm ee}$ directly and in closed form.

2. Explicit calculation of $\underline{\underline{D}}_{\rm ee}$

For the biaxial dielectric medium described by (7), $\underline{\underline{D}}_{\rm ee}$ can be obtained as a simplification of (5) in the form

$$\mathrm{i}\omega\epsilon_0 \underline{\underline{D}}_{\rm ee} = \frac{1}{4\pi}\int_{\phi=0}^{2\pi} \mathrm{d}\phi \int_{\theta=0}^{\pi} \mathrm{d}\theta\, \sin\theta \frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{\hat{\boldsymbol{x}}\cdot\underline{\underline{\hat{\epsilon}}}_{\rm bi}\cdot\hat{\boldsymbol{x}}}. \tag{8}$$

In the spirit of an earlier footnote, the product $\mathrm{i}\omega\epsilon_0\underline{\underline{D}}_{\rm ee}$ is dimensionless and thus directly comparable to definitions of the depolarization dyadic elsewhere [4]. Clearly, $\underline{\underline{D}}_{\rm ee}$ is an analytical function of the permittivities ϵ_x, ϵ_y, ϵ_z, as long as these (complex-valued) quantities do not lie on the negative real axis. In order to arrive at simple closed-form expressions for the components of $\underline{\underline{D}}_{\rm ee}$, it is temporarily assumed—purely for the benefit of the forthcoming integrations—that ϵ_x, ϵ_y, ϵ_z are real-valued and ordered in such a way that $\epsilon_z > \epsilon_y > \epsilon_x$. These restrictions to real and ordered parameters will be relaxed again in the final result.

In order to perform the ϕ-integration, a representation of $\hat{\boldsymbol{x}}$ is chosen in the following standard way: $\hat{\boldsymbol{x}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Consequently,

$$\hat{\boldsymbol{x}}\cdot\underline{\underline{\hat{\epsilon}}}_{\rm bi}\cdot\hat{\boldsymbol{x}} = (\epsilon_x\cos^2\phi + \epsilon_y\sin^2\phi)\sin^2\theta + \epsilon_z\cos^2\theta. \tag{9}$$

All off-diagonal terms in the dyadic $\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}$ in the numerator of (8) contain one of the three factors: $\sin\phi$, $\cos\phi$, $\sin\phi\cos\phi$, whereas the ϕ-dependence in the denominator is of the form $\sin^2\phi$ and $\cos^2\phi$. It is therefore not difficult to see that upon integration with respect to ϕ from 0 to 2π all the off-diagonal terms of $\underline{\underline{D}}_{\rm ee}$ vanish. The diagonal terms, on the other hand, provide integrals which can all be performed by making use of expression 2.562-1 in [14]

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{\gamma + \delta\sin^2\phi} = \frac{2\pi}{\sqrt{\gamma(\gamma+\delta)}} \tag{10}$$

for some parameters γ, δ, and valid for $\delta/\gamma > -1$. One obtains

$$\mathrm{i}\omega\epsilon_0\underline{\underline{D}}_{\rm ee} = \mathrm{diag}(D_x, D_y, D_z) \tag{11}$$

with the depolarization factors

$$D_x = \frac{1}{\epsilon_y - \epsilon_x}\left[-1 + \sqrt{\frac{\epsilon_z - \epsilon_y}{\epsilon_z - \epsilon_x}}\int_0^1 \mathrm{d}u\, \frac{\sqrt{\frac{\epsilon_y}{\epsilon_z-\epsilon_y} + u^2}}{\sqrt{\frac{\epsilon_x}{\epsilon_z-\epsilon_x} + u^2}}\right] \tag{12}$$

$$D_y = \frac{1}{\epsilon_y - \epsilon_x}\left[1 - \sqrt{\frac{\epsilon_z - \epsilon_x}{\epsilon_z - \epsilon_y}}\int_0^1 \mathrm{d}u\, \frac{\sqrt{\frac{\epsilon_x}{\epsilon_z-\epsilon_x} + u^2}}{\sqrt{\frac{\epsilon_y}{\epsilon_z-\epsilon_y} + u^2}}\right] \tag{13}$$

$$D_z = \frac{1}{\sqrt{\epsilon_z - \epsilon_x}\sqrt{\epsilon_z - \epsilon_y}}\int_0^1 \mathrm{d}u\, \frac{u^2}{\sqrt{\frac{\epsilon_y}{\epsilon_z-\epsilon_y} + u^2}\sqrt{\frac{\epsilon_x}{\epsilon_z-\epsilon_x} + u^2}} \tag{14}$$

where the new integration variable $u = \cos\theta$ was introduced for further simplification. It is noted parenthetically that at this stage the correct formulae for the uniaxial limit (in which $\epsilon_x = \epsilon_y$) are obtained by applying L'Hospital's rule in (12) and (13).

By using formulae 3.152-1 and 3.153-1 in [14] the integrals in (12)–(14) can be evaluated explicitly and one finds

$$D_x = \frac{\sqrt{\epsilon_y}}{(\epsilon_y - \epsilon_x)\sqrt{\epsilon_z - \epsilon_x}}[F(\alpha, q) - E(\alpha, q)] \tag{15}$$

$$D_y = \frac{1}{\epsilon_z - \epsilon_y}\left\{\frac{\epsilon_x - \epsilon_y}{\epsilon_z - \epsilon_y} - \sqrt{\frac{\epsilon_z - \epsilon_x}{\epsilon_y}}\left[\frac{\epsilon_x}{\epsilon_z - \epsilon_x}F(\alpha, q) - \frac{\epsilon_y}{\epsilon_z - \epsilon_y}E(\alpha, q)\right]\right\} \tag{16}$$

$$D_z = \frac{1}{\epsilon_z - \epsilon_y}\left[1 - \sqrt{\frac{\epsilon_y}{\epsilon_z - \epsilon_x}}E(\alpha, q)\right]. \tag{17}$$

In the above equations, $F(\alpha, q)$ is the elliptic integral of the first kind and $E(\alpha, q)$ is the elliptic integral of the second kind; their standard definitions being

$$F(\alpha, q) = \int_0^\alpha (1 - q^2\sin^2\lambda)^{-1/2}\,\mathrm{d}\lambda \tag{18}$$

$$E(\alpha, q) = \int_0^\alpha (1 - q^2\sin^2\lambda)^{1/2}\,\mathrm{d}\lambda \tag{19}$$

(see 8.111-2 and 8.111-3 of [14] respectively). The parameters appearing in the arguments of the elliptic functions are defined by

$$\alpha = \tan^{-1}\sqrt{\frac{\epsilon_z - \epsilon_x}{\epsilon_x}} \qquad q = \sqrt{\frac{\epsilon_z(\epsilon_y - \epsilon_x)}{\epsilon_y(\epsilon_z - \epsilon_x)}}. \tag{20}$$

The formulae (15)–(17) in conjunction with (11) provide the main result of this paper. It can be shown that in the uniaxial limit $\epsilon_x = \epsilon_y$ the expressions (15)–(17) reduce to those of the dielectric uniaxial case [8].

3. Discussion

Let $\boldsymbol{c}$ be an arbitrary vector and $\underline{\underline{A}}_{\mathrm{skew}}$ be a skew-symmetric dyadic: $\underline{\underline{A}}^T_{\mathrm{skew}} = -\underline{\underline{A}}_{\mathrm{skew}}$ (where ^{T} indicates transposition). Then, $\boldsymbol{c}\cdot\underline{\underline{A}}_{\mathrm{skew}}\cdot\boldsymbol{c} = 0$ and a glance at (5) shows that the depolarization dyadics, in general, are insensitive to any skew-symmetric parts contained within the constitutive dyadics. As a consequence, the results derived here for a biaxial dielectric medium remain unchanged for a medium characterized by

$$\underline{\underline{\epsilon}} = \underline{\underline{\epsilon}}_{\mathrm{bi}} + \underline{\underline{\epsilon}}_{\mathrm{skew}} \qquad \underline{\underline{\mu}} = \mu_0\underline{\underline{I}} + \underline{\underline{\mu}}_{\mathrm{skew}} \qquad \underline{\underline{\xi}} = \underline{\underline{\xi}}_{\mathrm{skew}} \qquad \underline{\underline{\zeta}} = \underline{\underline{\zeta}}_{\mathrm{skew}}. \tag{21}$$

It is also mentioned that $\underline{\underline{D}}_{\rm ee}$ as derived above for the biaxial medium is independent of the form of $\underline{\underline{\mu}}$; that is, $\underline{\underline{\mu}}$ can have a general anisotropy and need not be limited to isotropy, as was done here for simplicity of presentation. It is obvious, however, that in that particular case $\underline{\underline{D}}_{\rm mm}$ then becomes nontrivial.

A further generalization concerns the geometry: while results here have been presented for a spherical region, a more general expression than (5) was given in [6] for an ellipsoidal region. However, in the present context of a biaxial medium, this would only lead to a redefinition of the constitutive parameters ϵ_x, ϵ_y, ϵ_z, provided the ellipsoid's axes coincide with those of the chosen coordinate system.

Indeed, this argument leads immediately to the observation that the integral representation (5) can be subjected to an affine transformation. The result will be such that (5) turns from an integral representation for an anisotropic medium for a spherical region into one for an isotropic medium for an ellipsoidal region. For this interpretation it is crucial that $\epsilon_x''/\epsilon_x' = \epsilon_y''/\epsilon_y' = \epsilon_z''/\epsilon_z'$, with the prime denoting the real part and the double prime denoting the imaginary part. This condition is trivially fulfilled for real-valued permittivities. Such an approach was outlined elsewhere, see [15, 16] (and [17] for a comparative analysis). The affine transformation employed therein relies crucially on real-valued permittivities (or at least permittivities that fulfil the conditions of real/imaginary parts above). This technique permits the above-mentioned reduction of the problem to an ellipsoidal volume filled with an isotropic medium. Consequently, the depolarization factors of electrically small, ellipsoidal regions in vacuum—which have been known for many decades [18, 19]—can be straightforwardly used.

The consequence of assuming ϵ_x, ϵ_y and ϵ_z to be real-valued is that a *lossless* medium is considered. *Loss* manifests itself in the constitutive parameters becoming complex-valued. Because the depolarization dyadic $\underline{\underline{D}}_{\rm ee}$ is an analytical function of the permittivities ϵ_x, ϵ_y, ϵ_z, however, the closed-form expressions (15)–(17) here are valid also for complex values of ϵ_x, ϵ_y and ϵ_z, at least in the neighbourhood of the real axis. Therefore, in the present communication the emphasis has been on a direct derivation of the depolarization dyadic for a biaxial dielectric anisotropic medium without recourse to any affine or other transformation. The argument of analytic extension of the results into the regime of complex-valued constitutive parameters is considerably more transparent than the circuitous route of affine transformations. Further studies about the complex continuation of these results are currently under way; in particular, comparisons with numerical evaluations of (5) are being studied.

Generalization of the explicit evaluations presented here to fully bianisotropic media seems unfeasible at this time even though it was just seen that all skew-symmetric components of the constitutive dyadics are filtered out naturally. Unless the (bianisotropic) complexity of a medium is restricted to bi-isotropy or uniaxial bianisotropy (for which explicit results exist) the presence of the magnetoelectric dyadics $\underline{\underline{\xi}}$ and $\underline{\underline{\zeta}}$ in the integral representation (5) complicates the analysis considerably if not prohibitively. Both numerators and denominators of the integrals will then contain fourth powers of trigonometric functions in both ϕ and θ and even if possible, analytical manipulation of the integrals may simply be uneconomic when compared with numerical calculations.

Finally, as a bianisotropic medium is described by a large number of constitutive parameters, certain specific choices for these parameters may indeed provide the possibility of an explicit evaluation of (5). Such individual parameter adjustments or otherwise obscure parameter conditions miss the point, however. As important as the mathematical solvability is the physical realizability of a certain bianisotropic medium. Mathematically one is virtually free to choose the structure of the constitutive dyadics (provided certain basic consistency conditions are not violated). From a physical perspective, on the other hand, the macroscopic properties of a complex medium will depend fundamentally on its microscopic properties. As a consequence, one is in general not at liberty to change or adjust one specific constitutive parameter at the level of the macroscopic constitutive dyadics without affecting the remaining ones.

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